UNIVERSAL PRINCIPLES FOR KAZDAN-WARNER AND POHOZAEV-SCHOEN TYPE IDENTITIES

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Abstract. The classical Pohozaev identity constrains potential solutions of certain semilinear PDE boundary value problems. The Kazdan-Warner identity is a similar necessary condition important for the Nirenberg problem of conformally prescribing scalar curvature on the sphere. For dimensions $n \geq 3$ both identities are captured and extended by a single identity, due to Schoen in 1988. In each of the three cases the identity requires and involves an infinitesimal conformal symmetry. For structures with such a conformal vector field, we develop a very wide, and essentially complete, extension of this picture. Any conformally variational natural scalar invariant is shown to satisfy a Kazdan-Warner type identity, and a similar result holds for scalars that are the trace of a locally conserved 2-tensor. Scalars of the latter type are also seen to satisfy a Pohozaev-Schoen type identity on manifolds with boundary, and there are further extensions. These phenomena are explained and unified through the study of total and conformal variational theory, and in particular the gauge invariances of the functionals concerned. Our generalisation of the Pohozaev-Schoen identity is shown to be a complement to a standard conservation law from physics and general relativity.

1. Introduction

The current work is concerned with developing an effective and universal approach to treating and extending three identities, each of which plays a central role in the constraint of classes of non-linear geometric PDE problems.

Of these, the most easily stated arises in the problem of conformally prescribing scalar curvature; that is of determining, on a fixed conformal structure (M^n, c), which functions may be the scalar curvature Sc^g for some $g \in c$. This problem is especially interesting on the sphere, where it is known as the Nirenberg problem. While there are obvious constraints arising from the Gauss-Bonnet theorem, from the seminal work [33] of Kazdan-Warner it follows that there are positive functions on the 2-sphere S^2 that are not the curvatures of metrics that are pointwise conformal to the standard metric. A similar result was found in higher dimensions [34], and in all cases the results are a consequence of an identity satisfied by the first spherical harmonics. A well-known formulation and extension of these results is due to Bourguignon and Ezin [8], and is based around their identity: For any conformal (Killing) vector field X, of a closed Riemannian n-manifold (M, g), the

Key words and phrases. curvature prescription, Nirenberg problem, Kazdan-Warner identity, Pohozaev identity, conformal geometry, conservation laws.

scalar curvature satisfies

$$\int_M \mathcal{L}_X \operatorname{Sc}^g \; dv_g = 0, \tag{1}$$

where $\mathcal{L}_X$ denotes the Lie derivative.

Earlier Pohozaev described an identity which applies to, for example, star shaped manifolds M with smooth boundary ∂M in Euclidean space [42]. It was used, for example, to establish non-existence results for a class of semi-linear variants of eigenvalue boundary problems. These take the form $\Delta u + \lambda f(u) = 0$ with $u|_{\partial M} = 0$. Here f is a non-linear function that satisfies $f(0) = 0$. The Pohozaev identity states

$$\lambda n \int_M F(u) + \frac{2-n}{2}\lambda \int_M f(u)u = \frac{1}{2}\int_{\partial M} (x\cdot\nu)(\nabla_\nu u)^2; \tag{2}$$

x is the Euler vector field, ν is the outward unit normal, ∇_ν the directional derivative along ν, and $F(u) = \int_0^u f(t)dt$.

Remarkably the identities (1) and (2) are related. More precisely there is an identity due to Schoen [45, Proposition 1.4] which, at least for $n \geq 3$, includes both as special cases: For any conformal vector field X on a Riemannian n-manifold (M, g) with smooth boundary ∂M, the following identity holds

$$\int_M \mathcal{L}_X \operatorname{Sc} \; dv_g = \frac{2n}{n-2}\int_{\partial M} \left(\operatorname{Ric} - \frac{1}{n}\operatorname{Sc}\cdot g\right)(X,\nu) d\sigma_g; \tag{3}$$

here ν is the outward normal, and Ric denotes the Ricci curvature. This was proved using the Bianchi identities, and used as a balancing condition for approximate solutions to a PDE problem linked to the Yamabe equation. Since ∂M may be empty it is clear that (3) extends (1) for the cases $n \geq 3$. In Section 4.7 we describe, for the reader's convenience, how to recover (2) from (3).

The three identities have had a major impact in non-linear and geometric analysis, and are still used extensively in the current literature. This has motivated the development of analogous and related identities: For Kazdan-Warner type identities recovering or generalising (1) see for example [1], [7], [13], [15], [16], [28], [47]; for the Pohozaev identity (2) see [41], [44], [48]; and for Schoen's identity (3) (which is sometimes also referred to as a "Pohozaev identity") [17], [29]. This is by no means a complete list. Many of the works in the area treat specific curvature quantities, and are motivated by particular geometric problems. Exceptions include [13] which gives an analogue of (1) for all heat invariants corresponding to a conformally covariant operator. Most notably, by an an elegant and powerful argument, Bourguignon describes in [7] a very general framework for extending the "Kazdan-Warner identity" (1); this is further developed and applied by Delanoe and Robert in [16].

The identities and works mentioned suggest the following problems: For what scalar invariants $V = V(g)$ (replacing/generalising Sc^g) do we expect an analogue of the classical Kazdan-Warner identity (1)? Any such identity gives an immediate constraint for conformal curvature prescription on the sphere. Similarly, for what scalar invariants $V = V(g)$ do we expect an analogue of (3)? Note that this identity

gives a non-trivial constraint in a vastly wider range of geometric structures, so any extension has great potential for application. The third main problem is to precisely relate the two types of identity. For example if, in some general situation of closed manifolds, V satisfies an analogue of the Kazdan-Warner identity then do we expect it to also satisfy the Schoen identity (3) on manifolds with boundary? That there should be some subtlety here is clear from the factor of $\frac{1}{n-2}$ in (3); Schoen's construction apparently does not recover (1) in dimension 2.

In the current work we obtain essentially complete answers to the questions posed by showing that a closely related set of general principles underlie the Kazdan-Warner and Pohozaev-Schoen type identities. (Here we restrict to the case where X is a conformal vector field. There are clearly extensions to related settings, but this will be taken up elsewhere.) The principles involved are strongly related to the notion of symmetry and conservation that dates back to the work of D. Hilbert and E. Noether, and indeed this is our starting point in Section 2. Overall we obtain very general extensions of the Kazdan-Warner and Pohozaev-Schoen identities. Concerning the former, the main results are Theorems 2.7, Corollary 2.9, Theorem 2.11, and Theorem 2.14. The first three of these show that an identity of the type (1) is available for any natural scalar invariant which is conformally variational (as defined in Section 2.3) for suitable functionals of increasing generality; in each case the result is a direct consequence of symmetry invariance, or in other words of a gauge invariance, in the action functional concerned. The last Theorem 2.14 extends these results to show that in fact *any* conformally variational natural scalar satisfies such an identity. In this case the argument (cf. [7, 16]) is less direct and uses now the invariance of a 1-form on the space of metrics (in a conformal class), combined with the Lelong-Ferrand-Obata theory [36, 38].

The last mentioned approach appears to be necessary for a class of critical cases, but it misses the connection with the Schoen type identities (3). On the other hand the very simple argument behind Theorem 2.7 involves specialising total metric variations, and so is linked to locally conserved 2-tensors (as explained in Section 2.3). Through this its proof is intimately connected to Theorem 3.1 which extends (3) to an identity that holds for the trace and trace-free parts of any locally conserved 2-tensor. This is a very large class of invariants that need not be natural (see e.g. Section 2.2). It is precisely the difference between Theorem 2.7 (or Theorem 3.1) and Theorem 2.14 that is behind the $\frac{1}{n-2}$ factor mentioned earlier, and the generalisation of this phenomenon. See also Corollary 3.3 and the discussion below.

In Section 3.2 we show that the generalised Schoen identity of Theorem 3.1 is a precise complement to the usual conservation theory extant in the Physics literature.

The main results mentioned, and their proofs, appear to unify, simplify, and considerably extend most of the existing related results in the literature; see Section 4 where we show a number of new results, as well the simplification and unification of a number of recent particular results in the literature. Specific examples treated include Gauss-Bonnet curvatures, Q-curvatures, renormalised volume coefficients,

and the mean curvature of a conformal immersion. Although we do not directly discuss extensions of the Pohozaev identity 2 it is clear that such can be obtained from Theorem 3.1 by, for example, an analogue of treatment in Section 4.7.

ARG would like to thank Alice Chang, Paul Yang, Matt Gursky and Frédéric Rochon for useful discussions at the meeting "Geometric and Nonlinear Partial Differential Equations" held at Mission Beach Resort, Queensland, 2010. A draft of this work was presented there. Discussions with Robin Graham and Andreas Juhl at the Tambara Institute of Mathematics worskop " Parabolic Geometries and Related Topics I", November 2010, are also much appreciated. In particular Graham provided an answer to a question posed in the first draft, see Theorem 4.6.

2. The Hilbert-Noether identities for gradients

Until further notice we shall suppose we work on a closed (compact without boundary) oriented connected manifold M, of dimension $n \geq 2$ and usually equipped with a Riemannian metric g. However we also consider the space $\mathcal{M}$ of such metrics on M, that space equipped with the compact open C^∞ topology. For simplicity all structures and sections throughout shall be considered smooth (C^∞).

A real valued functional $\mathcal{S}$ on $\mathcal{M}$ is called a *Riemannian functional* if it is diffeomorphism invariant in the sense that it satisfies

$$\mathcal{S}(\varphi^* g) = \mathcal{S}(g), \tag{4}$$

for all $g \in \mathcal{M}$, and for all diffeomorphisms $\varphi : M \to M$.

A *natural scalar (Riemannian) invariant* (see e.g. [46]) is a scalar valued function which is given by a universal expression, which is polynomial in the finite jets of the metric and its inverse, and which has the property that for any diffeomorphism $\varphi : M \to M$ we have

$$\varphi^* L(g) = L(\varphi^* g). \tag{5}$$

An important class of Riemannian functionals, and our main (though certainly not exclusive) focus here, arise from the integral of such Lagrangians: that is $g \mapsto \mathcal{S}(g)$ where

$$\mathcal{S}(g) = \int_M L(g) dv_g \tag{6}$$

where dv_g is the metric measure.

Remark 2.1. *One may construct natural invariants in an obvious way by complete contractions, using the metric, its inverse, and the volume form, of expressions polynomial in the Riemann curvature, and its Levi-Civita covariant derivatives. In fact all natural invariants arise this way as follows by a well known argument using Weyl's classical invariant theory and Riemann normal coordinates, see e.g.* [2].

2.1. **Total metric variations.** The tangent space to $\mathcal{M}$ is naturally identified with the (smooth) section space of S^2M. A differentiable Riemannian functional is said to have a *gradient* $B(g)$ at g, if $B(g)$ is a smooth section of S^2M and, for all $h \in \Gamma(S^2M)$,

$$\mathcal{S}'(g)(h) = \int_M (h, B(g)) dv_g \tag{7}$$

where $(\cdot,\cdot)$ denotes the local pairing of tensors given by metric contraction. In an abstract index notation we shall write B_{ab} for $B(g)$.

If we specialise now to h arising from the pullback along a diffeomorphism generated by a vector field X, then $h = \mathcal{L}_X g$ and we have

$$0 = \int_M (\mathcal{L}_X g, B(g)) dv_g, \tag{8}$$

from the diffeomorphism invariance of the Riemannian functional $\mathcal{S}$. In terms of the Levi-Civita connection ∇ (for g), we have $(\mathcal{L}_X g)_{ab} = 2\nabla_{(a}X_{b)}$. Thus integrating by parts in (8) we see that

$$0 = \int_M X^b \nabla^a B_{ab} dv_g.$$

Since the X^b is arbitrary we conclude

$$\nabla^a B_{ab} = 0, \tag{9}$$

and we shall say that B is *locally conserved.* This is a standard identity for the gradient of a Riemannian functional, and is attributed to Hilbert [6, 31]. Identities derived from symmetries or "gauge invariance", such as this, are often called Noether identities in the literature.

Now we consider the case where $\mathcal{S}(g)$ is given by a natural Lagrangian, as in (6). It follows from the result mentioned in Remark 2.1, and integration by parts, that each directional derivative of $\mathcal{S}(g)$ is of the form (7) where $B(g)$ is a natural (tensor-valued) invariant. From this in turn we conclude that $\mathcal{S}(g)$ is differentiable and so the above discussion applies immediately; in particular the natural tensor $B = B(g)$ satisfies (9).

Remark 2.2. *Although the argument above has used a compact Riemannian setting, as an aside here we note the following: since B is given by a universal expression in terms of the Riemannian curvature and its covariant derivatives, it follows that the local result (9) holds on any manifold and in any signature.*

To see how non-trivial results may arise from the diffeomorphism invariance of an action it is useful to understand, via an infinitesimal argument, how the gradient is generated in the case of a natural Lagrangian function $L = L(g)$. Consider a curve of metrics g^t through $g = g^0$. Calculating the derivative of (6) at $t = 0$ involves computing the linearisation of $L(g)$ (at g),

$$L'(h) := \frac{d}{dt}\Big|_{t=0} L(g^t),$$

and also the contribution from the measure:

$$\frac{d}{dt}\Big|_{t=0} dv_{g^t} = \frac{1}{2} g^{ab} h_{ab} dv_g.$$

Putting these together we have

$$\frac{d}{dt}\Big|_{t=0} \mathcal{S}(g^t) = \int_M \big(L'(h) + \frac{1}{2} L(g) g^{ab} h_{ab}\big) dv_g. \tag{10}$$

However for h arising from an infinitesimal diffeomorphism we have, as mentioned, $h = \mathcal{L}_X g$. Thus $\frac{1}{2} g^{ab} h_{ab} = \nabla_a X^a = \operatorname{div} X$. On the other hand the infinitesimal version of the naturality condition (5) is

$$L'(\mathcal{L}_X g) = \mathcal{L}_X L(g) \tag{11}$$

and so for $h = \mathcal{L}_X g$ we have $(L'(h) + \frac{1}{2} L(g) g^{ab} h_{ab}) = \operatorname{div}(L(g)X)$ whence the right hand side of (10) is zero. The non-trivial identity (7) arises by calculating in another order. We first integrate (10) by parts to yield (7), and then proceed as argued earlier. So the information contained in the difference between the two ways of calculating arises entirely from (11).

2.2. **Generalised energy-momentum tensors.** The local conservation of natural gradients is a unifying feature in the discussion which follows. In fact, as we shall see, a broader class of gradients satisfy (9). Suppose that rather than restrict to L being a natural scalar invariant of (M, g), we allow L as follows. We assume L is a scalar valued function which is given by a universal expression, which is polynomial in the finite jets of the metric and its inverse, and also in the finite jets of a collection of other fields that we shall collectively denote Ψ (and regard as a single field). So we may write $L = L(g, \Psi)$. The fields that make up Ψ may be tensor fields, but also could include for example connections. We shall not be concerned with the details; it is rather naturality in this context that is important. We shall insist that L satisfies

$$\varphi^* L(g, \Psi) = L(\varphi^* g, \varphi^* \Psi) \tag{12}$$

for any diffeomorphism $\varphi : M \to M$. So certainly we require that the nature of the fields Ψ is such that their pullback under diffeomorphism makes sense, but this is a very weak restriction. We shall call such $L(g, \Psi)$ *coupled scalar invariants.*

Now we assume that

$$\mathcal{S}(g, \Psi) := \int_M L(g, \Psi) dv_g$$

is separately Frechet differentiable with respect to g and Ψ, and that there are respective partial gradients $B(g, \Psi)$, $E(g, \Psi)$, satisfying

$$(D_1 \mathcal{S}(g, \Psi))(h) = \int_M (h, B(g, \Psi)) dv_g$$

and

$$(D_2 \mathcal{S}(g, \Psi))(h) = \int_M \langle \psi, E(g, \Psi) \rangle dv_g$$

where ψ is in the formal tangent space at Ψ to the field (system) Ψ and $\langle\cdot,\cdot\rangle$ is the pointwise dual pairing that arises naturally in the problem. (In the other display the notation is as in (7).) We shall refer to $B(g,\Psi)$ as the metric gradient.

The equation $E(g,\Psi)=0$ is a generalised Euler-Lagrange system. We have the following result.

Theorem 2.3. *On (M,g), let Ψ_0 be a solution of the generalised Euler-Lagrange system*

$$E(g,\Psi)=0.$$

Then the metric gradient $B(g,\Psi_0)$ is locally conserved, that is

$$\nabla^a B_{ab}(g,\Psi_0)=0. \tag{13}$$

Proof. From (12) it follows that $S(g,\Psi)$ is diffeomorphism invariant. Thus, differentiating $S(g,\Psi)$ along the pullback of an infinitesimal diffeomorphism generated by a vector field X, and using the chain and product rule under the integral, we have a generalisation of (8), viz.

$$0=\int_M \Big((\mathcal{L}_X g, B(g,\Psi)) + \langle \mathcal{L}_X \Psi, E(g,\Psi)\rangle \Big) dv_g,$$

where the derivative of (12) is used. Thus if we calculate along Ψ_0 satisfying $E(g,\Psi_0)=0$, then this reduces to $0=\int_M(\mathcal{L}_X g, B(g,\Psi))dv_g$ and we argue as below (8) to conclude (13). $\square$

Remark 2.4. *The argument above is a minor variant of that in* [30]*, which treats the case that L depends on at most first covariant derivatives of Ψ. In that setting $E(g,\Psi_0)=0$ gives the standard Euler-Lagrange equations of continuum mechanics and they term $B(g,\Psi_0)$ an "energy-momentum tensor". In certain contexts the same $B(g,\Psi_0)$ is sometimes termed a stress-energy tensor* [3, 4].

2.3. **Conformal variations.** On a manifold M, a natural scalar invariant V is said to be *conformally variational* within a conformal class of metrics $\mathcal{C}=\{\widehat{g}=e^{2\Upsilon}g \mid \Upsilon\in C^\infty(M)\}$ if there is a functional $\mathcal{S}(g)$ on $\mathcal{C}$ with

$$\mathcal{S}^\bullet(g)(\omega)=2\int_M \omega V\, dv_g\,, \qquad \text{all } \omega\in C^\infty(M). \tag{14}$$

As above dv_g is the Riemannian measure, and here

$$\mathcal{S}^\bullet(g)(\omega):=\frac{d}{ds}\Big|_{s=0}\mathcal{S}(e^{2s\omega}g). \tag{15}$$

In (15), the curve of metrics $e^{2s\omega}g$ may be replaced by any curve with the same initial tangent $g^\bullet=2\omega g$. The property of being variational can depend both on L, and on the conformal class $\mathcal{C}$.

We shall consider first two important cases, with the first case as follows.

Definition 2.5. *We shall say that V, a natural scalar invariant, is* <u>*naturally*</u> *conformally variational if it arises as in (14) above from a Riemannian* <u>*functional*</u> *$\mathcal{S}$ that admits a gradient (as in (7)) for any $g\in\mathcal{C}$.*

Suppose now $\mathcal{S}$ is as in Definition 2.5 and we calculate (15) via a specialisation of the total metric variation computation (7). It follows that

$$\mathcal{S}^\bullet(g)(\omega) = 2\int_M (\omega g, B)\, dv_g \tag{16}$$

whence, in particular, $V = g^{ab}B_{ab}$. We summarise this observation.

Lemma 2.6. *If $\mathcal{S}$ is a Riemannian functional with gradient B_{ab} at g, then the function V in (14) is given by $g^{ab}B_{ab}$.*

Recall that for total metric variations the key integral relation underlying the Hilbert-Noether identity is (8). Comparing this with (16) we see that, in the restricted setting of conformal variations, (8) still yields constraints provided $\mathcal{L}_X g = 2\omega g$. But this exactly means that X is a conformal vector field and $\omega = \frac{1}{n}\operatorname{div} X$. Then (8) states

$$0 = \int_M (\operatorname{div} X) V\, dv_g .$$

So, integrating by parts, we have the following.

Theorem 2.7. *If V is naturally conformally variational, then for any conformal vector field X^a on a closed Riemannian manifold (M, g), we have*

$$0 = \int_M (\operatorname{div} X) V\, dv_g = -\int_M (\mathcal{L}_X V) dv_g . \tag{17}$$

One might suppose that Definition 2.5, as used in Theorem 2.7, is restrictive. In fact in most cases it is not. To make this precise we need a further definition. A natural invariant L (possibly tensor valued) is said to have *weight ℓ* if uniform dilation of the metric has the effect $L[A^2 g] = A^\ell L[g]$ for all $0 < A \in \mathbb{R}$. For example, the scalar curvature has weight -2. It is not essentially restrictive to consider only invariants of a well defined weight, since it is easily shown that any natural scalar invariant is a sum of such. The key to the claim that began this paragraph is the following result.

Proposition 2.8. [11] *If V, of weight $\ell \neq -n$, is a conformally variational local scalar invariant on a closed Riemannian conformal n-manifold $(M, \mathcal{C})$, then*

$$\mathcal{S}(g) := (n+\ell)^{-1}\int_M V\, dv_g \tag{18}$$

is a Riemannian functional for V in $\mathcal{C}$; that is (14) holds.

Now by the discussion of natural Lagrangians in Section 2.1, it follows that (7) holds for $\mathcal{S}$ as in (18), and so $\mathcal{S}$ satisfies Definition 2.5. Thus we have the following.

Corollary 2.9. *On a closed Riemannian n-manifold a natural scalar invariant V, of weight $\ell \neq -n$, is conformally variational if and only if it is naturally conformally variational.*

The scalar curvature is well known to be conformally variational and so Theorem 2.7 certainly extends the results of Bourguignon-Ezin [8] for the scalar curvature

in dimensions $n \geq 3$. In fact conformally variational invariants are not at all rare, and so the extension is vast; we shall take up this point in Section 4.

Next we show that a slight variant of the above also recovers and extends the identity from [8, 33] for the Gauss curvature in dimension 2. Above we used that it is insightful to use the gradient B when this is available. That observation will also be critical in the next section. However the existence of a total metric variation gradient, as in (7), is not necessary to see a Kazdan-Warner type identity arise from gauge invariance.

Definition 2.10. *We shall say that V, a natural scalar invariant, is normally conformally variational if it arises via (14) with $\mathcal{S}$ a Riemannian functional.*

Note that this is a strictly broader class of invariants than above: if V is naturally conformally variational then it is normally conformally variational.

Theorem 2.11. *The identity (17) holds if we assume only that V is normally conformally variational (with also the other conditions of Theorem 2.7 imposed).*

Proof. We follow the idea of Section 2.1, but restrict at the outset to the case that X is a conformal vector field. Again from the diffeomorphism invariance of the Riemannian functional we have $0 = \mathcal{S}'(g)(h)$ where $h = \mathcal{L}_X g$. But h is a conformal variation: $h = \frac{2}{n}(\operatorname{div} X)g$. So $\mathcal{S}'(g)(h) = \mathcal{S}^\bullet(g)(\frac{1}{n}\operatorname{div} X)$ and since, by assumption, V and $\mathcal{S}$ are related by (14) the result follows. $\square$

Example 2.12. *On a closed Riemannian 2-manifold (M, g) if we take $\mathcal{S}(g) :=$ $\det \Delta_g / A(g)$, where $A(g)$ is the total area and $\det \Delta_g$ is the functional determinant of the Laplace-Beltrami operator, then there is the Polyakov formula* [39, 43] *for conformal variation*

$$\mathcal{S}^\bullet(g)(\omega) = c \cdot \int_M \omega Q dv_g$$

where Q is the Gauss curvature and $c \neq 0$ is a constant. $\mathcal{S}(g)$ is a Riemannian functional and so we conclude from Theorem 2.11 that for any conformal vector field X on M we have $\int_M \mathcal{L}_X Q \, dv_g = 0$.

Remark 2.13. *In view of the derivations in Theorems 2.7 and 2.11 it is clear that the Kazdan-Warner identities are related to Noether-Hilbert principles. Note here we do not expect an analogue of (9): The result here is necessarily global, since the common ground between (8) and (16) involves conformal vector fields which are global objects.*

From the proof of Theorem 2.11 it is evident that we may obtain an identity at a particular $g_1 \in \mathcal{C}$ without the full force of (4). Indeed we simply need $\mathcal{S}'(g_1)(h) = 0$ where h is $\mathcal{L}_X g_1$ and X a conformal vector field. If $V(g)$ is a conformally variational natural invariant this is achieved by the functional $\mathcal{S}(g) = \int_M \omega V(g) dv_g$ on C, where $g = e^{2\omega} g_1$, $\omega \in C^\infty(M)$. This follows from the following argument, which is a trivial adaption of a result from [7, 16].

Theorem 2.14. *Suppose that X is a conformal vector field on a closed Riemannian conformal manifold $(M, \mathcal{C})$, and that $V = V(g)$ is a conformally variational*

natural scalar invariant. Then

$$\int_M (\operatorname{div} X) V(g)\ dv_g.$$

is independent of the choice of metric $g \in \mathcal{C}$, and hence is zero.

Proof. Fix any metric $g_0 \in \mathcal{C}$. If V is conformally variational then the linearisation of the map $\omega \mapsto V(e^{2\omega} g_0)$, $\omega \in C^\infty(M)$, is formally-self-adjoint (see e.g. [11]). Identifying $C^\infty(M)$ with the tangent space to $\mathcal{C}$, it follows that the 1-form on $\mathcal{C}$

$$C^\infty(M) \ni \omega \mapsto \int_M \omega V(g)\ dv_g$$

is closed [7, 11]. Now suppose that $\tilde{X}$ is the vector field on $\mathcal{C}$ induced by a conformal diffeomorphism X on M. From the diffeomorphism invariance of this 1-form it is annihilated by $\mathcal{L}_{\tilde{X}}$. Then using the Cartan formula $\mathcal{L}_{\tilde{X}} = d\iota_{\tilde{X}} + \iota_{\tilde{X}} d$, and the identification of $\tilde{X}$ with $\frac{1}{n} \operatorname{div} X$, it follows that $\int_M (\operatorname{div} X) V(g)\ dv_g$ is constant on $\mathcal{C}$ as claimed.

It follows that if there is a metric $g_0 \in \mathcal{C}$ such that $V(g_0)$ is constant then for any metric $g \in \mathcal{C}$ we have

$$\int_M \mathcal{L}_X V\ dv_g = 0.$$

In particular this holds on the sphere S^n with its standard conformal structure. However by the Lelong-Ferrand-Obata theorem [36, 38] if M is any other conformal manifold then X is necessarily a Killing vector field. In that case we have $\mathcal{L}_X V = \operatorname{div}(VX)$ and so $\int_M \mathcal{L}_X V\ dv_g = 0$. $\square$

Remark 2.15. *While this Theorem gives the strongest result, it uses a less direct argument than that of Theorems 2.7 and 2.11, and this argument partly loses contact with the Hilbert-Noether principles, and in most cases is not necessary (as follows from Corollary 2.9). Most importantly, as we shall see below, the proof of Theorem 2.7 naturally suggests, and links it to, a generalisation of the Schoen identity.*

On the other hand for natural invariants of weight $-n$ we expect to need stronger arguments: for example if V is a conformal covariant of weight $-n$, then $V dv_g$ is a conformally invariant n-form and so $\int_M V dv_g$ is conformally invariant. It is easily seen that such a matching of weights between dv_g and V causes a breakdown in the argument of Theorem 2.7.

The result in [16] *corresponding to Theorem 2.14 uses that the linearisation of $\omega \mapsto V(e^{2\omega} g)$ is formally self adjoint, without any explicit mention that V is variational. But for a natural scalar invariant this self-adjointness condition is equivalent to it being conformally variational, as follows from a trivial variant of* [11, Lemma 2(ii)].

3. Manifolds with boundary and conservation

Let M be Riemannian manifold with boundary ∂M. To avoid unnecessary restriction we allow here the possibility that ∂M is the empty set. In this setting, and using a different approach to the above, we derive a result that strictly generalises Theorem 2.7 and the Schoen identity (3).

3.1. **A generalisation of the Schoen identity.** On M, let B be a symmetric 2-tensor with compact support, and X any tangent vector field. Then by the Gauss formula for Stokes' Theorem,

$$\int_M \nabla^a(B_{ab}X^b)dv_g = \int_{\partial M} B_{ab}X^a\nu^b d\sigma_g,$$

where ν and $d\sigma_g$ are, respectively, the outward unit normal and the induced metric measure along ∂M.

Now if the tensor B is locally conserved, meaning that $\nabla^a B_{ab} = 0$, then

$$2\nabla^a(B_{ab}X^b) = 2B_{ab}\nabla^a X^b = (B, \mathcal{L}_X g). \tag{19}$$

In particular if X is a conformal vector field then

$$\nabla^a(B_{ab}X^b) = \frac{1}{n}V \operatorname{div} X$$

where V is the metric trace of B, i.e. $V := g^{ab}B_{ab}$, and $\operatorname{div} X = \nabla_a X^a$. So

$$n\int_{\partial M} B_{ab}X^a\nu^b d\sigma_g = \int_M V(\operatorname{div} X)dv_g\ . \tag{20}$$

A related identity arises from the (metric) trace-free part of B_{ab}, that is

$$\overset{o}{B}_{ab} := B_{ab} - \frac{1}{n}g_{ab}V.$$

If X is a conformal Killing vector field then

$$\nabla^a(\overset{o}{B}_{ab}X^b) = (\nabla^a\overset{o}{B}_{ab})X^b + \overset{o}{B}_{ab}\nabla^a X^b.$$

But then $\overset{o}{B}_{ab}\nabla^a X^b = 0$, since $\overset{o}{B}_{ab}$ is symmetric trace-free, while $\frac{1}{2}(\nabla^a X^b + (\nabla^b X^a) = \frac{1}{n}g^{ab}\operatorname{div} X$. For the other term observe that

$$X^b\nabla^a\overset{o}{B}_{ab} = X^b(\nabla^a B_{ab} - \frac{1}{n}\nabla_b V) = -\frac{1}{n}X^b\nabla_b V.$$

Thus

$$\begin{aligned} \textstyle\int_M \mathcal{L}_X V\ dv_g &= -n\textstyle\int_M \nabla^a(\overset{o}{B}_{ab}X^b)dv_g \\ &= -n\textstyle\int_{\partial M} \overset{o}{B}_{ab}X^a\nu^b d\sigma_g \end{aligned}$$

Recalling also (9), we summarise as follows.

Theorem 3.1. *On an oriented Riemannian manifold M with boundary ∂M the following holds. If B is a locally conserved symmetric 2-tensor, of compact support, and X is a conformal vector field, then*

$$\int_M \mathcal{L}_X V\ dv_g = -n\int_{\partial M} \overset{o}{B}_{ab}X^a\nu^b d\sigma_g, \tag{21}$$

where V is the metric trace of B, i.e. $V = g^{ab}B_{ab}$. In particular this holds for any gradient tensor or generalised energy-momentum tensor B that has compact support.

In particular the above applies when $\partial M = \emptyset$. Thus we have another Kazdan-Warner type result. For emphasis we state this specialisation.

Corollary 3.2. *On a Riemannian manifold M, without boundary, let V be the metric trace of a compactly supported and locally conserved symmetric 2-tensor B. Then for any conformal vector field X we have*

$$\int_M \mathcal{L}_X V dv_g = 0.$$

Using Proposition 2.8 and the result (16), Theorem 3.1 also gives the following result.

Corollary 3.3. *Suppose that the natural scalar invariant V is naturally conformally variational on a compact n-manifold with boundary. Then $V = g^{ab}B_{ab}$ where B_{ab} is a natural gradient of some Riemannian functional, and the relation (21) holds for any conformal vector field X. If V has a well defined weight $\ell \neq -n$, then B is gradient of the functional*

$$\mathcal{S}(g) = \frac{1}{n+\ell}\int_M V dv_g.$$

A special case of Theorem 3.1 arises when B_{ab} is (a non-zero multiple of) the *Einstein tensor*

$$B_{ab} := P_{ab} - g_{ab}J,$$

which is the gradient arising from the Einstein-Hilbert action; here $n \geq 3$ and we assume compact support. So then $V = (1-n)J$. Here P_{ab} is the Schouten tensor and $J = g^{ab}P_{ab}$; in terms of the Ricci and scalar curvatures, this is characterised by

$$\mathrm{Ric}_{ab} = (n-2)P_{ab} + Jg_{ab},$$

whence $\mathrm{Sc} = 2(n-1)J$. So then ($2\times$) (21) states

$$2(1-n)\int_M \mathcal{L}_X J \; dv_g = -2n\int_{\partial M} P_{(ab)_0} X^a \nu^b d\sigma_g,$$

where $(\cdots)_0$ indicates the trace-free symmetric part. In other terms we obtain,

$$\int_M \mathcal{L}_X \,\mathrm{Sc}\, dv_g = \frac{2n}{n-2}\int_{\partial M} \mathrm{Ric}_{(ab)_0}\, X^a \nu^b d\sigma_g,$$

as a special case of Theorem 3.1. This is precisely the Schoen identity (3) from the introduction.

Remark 3.4. *The identity (20) is widely used in the literature, see e.g.* [3, 4, 5, 17, 40] *and references therein.*

Remark 3.5. *Theorem 3.1 produces a Schoen-type identity for every locally conserved symmetric 2-tensor, and thus in particular for every natural gradient, or generalised energy-momentum tensor. The surprising aspect of the Theorem is that it provides a rather subtle global relation between the trace and trace-free parts of a locally conserved 2-tensor.*

3.2. **Conserved quantities.** A Killing vector field X is of course also a conformal Killing vector field. However Theorem 2.7 and Corollary 3.2 are vacuous for such X: if X is a Killing vector then for any function f on a closed Riemannian manifold M we have $\int_M \mathcal{L}_X f \; dv_g = 0$, since $\operatorname{div} X = 0$ and so $\mathcal{L}_X f = \operatorname{div}(fX)$. In both Theorem 2.7 and Corollary 3.2 the function V is the trace of a locally conserved symmetric 2-tensor B. Thus these results are also obviously vacuous if in fact B is trace-free, so $B = \overset{o}{B}$, even if X is not Killing.

It is natural to ask of the meaning of the corresponding Pohozaev-Schoen type identities in these degenerate cases. This brings us to the following result, at least part of which is well known in the physics literature (see e.g. [30]).

Proposition 3.6. *Suppose that X is a (conformal) Killing vector field on a Riemannian manifold, and B is a locally conserved (metric trace-free) symmetric 2-tensor. Then the corresponding current $J_a := B_{ab}X^b$ is locally conserved, that is*

$$\operatorname{div} J = 0. \tag{22}$$

Proof. If B is a symmetric 2-tensor that satisfies $\nabla^a B_{ab} = 0$ then for any vector field X^b, and setting $J_a := B_{ab}X^b$, it follows immediately from (19) that $\nabla_a J^a$ is zero if and only if $\mathcal{L}_X g$ is pointwise orthogonal to B. Thus in particular if $\mathcal{L}_X g = 0$ this holds. It also holds if instead $\mathcal{L}_X g = \frac{2}{n}(\operatorname{div} X)g$, provided B its trace-free. □

For either of the cases in the Proposition, it is easily seen that the Pohozaev-Schoen type identity of Theorem 3.1 is equivalent to the usual flux conservation law for conserved currents.

On the other hand if B is locally conserved but not necessarily trace-free then from (19) we have, in our current notation, $\operatorname{div} J = V \operatorname{div} X/n$, for a conformal Killing vector field X. Then on the left-hand-side of (20) $\int_{\partial M} B_{ab}X^a\nu^b \; d\sigma_g = \int_{\partial M} J_a\nu^a \; d\sigma_g$ is a measure of flux reflecting conservation failure.

Thus we see that the identity of Theorem 3.1 is exactly a complement of the usual conservation law for conserved currents. To underscore this point we note here that the Proposition above provides a route to proliferating conserved quantities on geometries with symmetry.

Theorem 3.7. *Each natural scalar invariant L determines a corresponding natural gradient*

$$B^L_{ab}, \tag{23}$$

and so the following:

- *On any Riemannian manifold with a Killing vector field X one obtains a corresponding canonical and locally conserved current J^L_a, (i.e. J^L satisfies (22)).*

- *If L has the property that, on closed manifolds, $\mathcal{S}(g) = \int_M L dv_g$ is conformally invariant, then B^L_{ab} is conformally covariant and trace-free. It follows that on any Riemannian manifold with a conformal Killing vector field X the corresponding canonical and locally conserved current J^L_a is conformally covariant. In this case the local conservation equation (22) is conformally invariant.*
- *In either case L determines a non-local invariant*
$$I^L_\Sigma := \int_\Sigma J^L_a d\sigma^a,$$
for each hypersurface Σ, with the property that $I^L_{\Sigma_1} = I^L_{\Sigma_2}$ if Σ_1 and Σ_2 are homologous hypersurfaces sharing the same boundary. If $\mathcal{S}(g) = \int_M L dv_g$ is conformally invariant then I^L_Σ is conformally invariant.

Proof. We observed in section 2.1 that on closed manifolds the action determined by L, viz. $S(g) = \int_M L\ dv_g$ has a corresponding natural gradient B^L_{ab}, and this is locally conserved, cf. (9). Then, as a natural tensor, B^L_{ab} is given by a universal formula in terms of partial (metric or volume form) contractions of Levi-Civita covariant derivatives of the Riemannian curvature. We now take this universal formula as defining the symmetric and locally conserved tensor B^L_{ab}.

Thus the first result then follows from Proposition 3.6 with

$$J^L_a := B^L_{ab} X^b. \tag{24}$$

Now set $V := g^{ab} B^L_{ab}$. If $\mathcal{S}(g)$ is conformally invariant (on closed manifolds) then (7) must be zero when, for example $g^t = e^{2t\omega} g$, $\omega \in C^\infty(M)$. But in this case $h = 2\omega g$, so

$$\int_M \omega V\ dv_g = 0.$$

This must hold for arbitrary $\omega \in C^\infty(M)$, and so $V = 0$, i.e. B^L is metric trace-free. Again this must be also true of the universal formula for B^L.

Thus the claim that J^L (as in (24) with X now conformal Killing) is conserved, as stated in the second point of the Theorem, also follows from Proposition 3.6. An easy argument involving second variations of $\mathcal{S}(g)$, that mix conformal and total metric variations, then shows that B^L is necessarily conformally invariant (see e.g. [10] where also the notion of conformal invariance, as used here, is discussed).

Finally for the second point, if L has a well-defined weight (and any L is a sum of such) then $\mathcal{S}(g)$ conformally invariant and non-trivial implies this weight is $-n$. Since any natural scalar is a sum of invariants each of which has a well-defined weights, it follows that we may assume without loss of generality that L has weight $-n$. It follows that B^L, and hence also J^L, has weight $2-n$ and in fact they are then conformally covariant of weight $2-n$. In this case it is well known (and easily verified) that the equation (22) is conformally invariant.

The third point is then immediate from the divergence theorem, save for the comment about conformal invariance. But the latter is an easy consequence of the weight of J^L and its conformal covariance.

□

Remark 3.8. *If L is a coupled scalar invariant, in the sense of Section 2.2, then we may replace B^L, In Theorem 3.7, by the corresponding generalised energy-momentum tensor.*

3.3. **Other signatures.** For simplicity of exposition in the above we have restricted to Riemannian signature. In fact all results above in Section 3 extend as stated to pseudo-Riemannian manifolds of any signature with the following restrictions and minor adjustments: the boundary conormal ν_a is nowhere null; it is normalised so that $g_{ab}\nu^a\nu^b = \pm 1$; and it satisfies that at any point of the boundary $\nu_a X^a$ is positive if X^a is an outward pointing tangent vector.

The restriction that ν_a be nowhere null can be removed if statements are adjusted appropriately. We leave this to the reader.

4. Examples and Applications

Theorems 2.7 and 3.1 are already very general. For example begin with *any* natural scalar invariant L. Since natural scalar invariants are easily written down using Weyl's classical invariant theorem [2], we may readily proliferate examples. Then generically the total metric variation of $\mathcal{S}(g) := \int_M L(g) dv_g$ will yield a corresponding non-trivial Euler-Lagrange tensor B^L via (7). Exceptions are those natural scalars L whose integral is a topological (or smooth structure) invariant, such as the Pfaffian in even dimensions. In any case of $B^L \neq 0$ Theorem 3.1 is non-trivial.

If the integral of L is conformally invariant (so the weight of L is $-n$) then B^L is trace-free (by Theorem 3.7), and the left-hand-side of (21) vanishes in Theorem 3.1; the latter nevertheless yielding a non-trivial constraint as discussed in Section 3.2. Otherwise, by (16) we see that $V^L := \mathrm{tr}^g(B^L)$ is a conformally variational natural scalar and Theorem 2.7 and Theorem 3.1 apply non-trivially.

Suppose that V^L has a well defined weight $\ell \neq -n$ (as follows if L does). Then the map

$$L \to (n+\ell)^{-1} V^L$$

may be regarded as a projection to the *conformally variational part* of L, as follows from Proposition 2.8. Ignoring possible deeper applications, this at least shows that conformally variational scalar invariants are, in a suitable sense, extremely common. We discuss some cases below.

Note that if V is a weight $\ell \neq -n$ scalar invariant, then it being conformally variational immediately implies it has some properties which are analogous to the scalar curvature. In particular we have the following. Let us write B^V_{ab} for the gradient of the functional $\mathcal{S}^V(g) := (n+\ell)^{-1} \int_M V dv_g$. Then $V = g^{ab} B^V_{ab}$ and, denoting by $\mathring{B}^V$ the trace-free part of B^V, we have this observation:

Proposition 4.1. *If $\mathring{B}^V_{ab} = 0$ then $V =$ constant.*

Proof. This is an immediate consequence of $\nabla^a B^V_{ab} = 0$. $\square$

The point is that in the case of V being the scalar curvature $\mathring{B}^V_{ab} = 0$ expresses the Einstein equations. In that setting the result in the Proposition is often viewed as

a consequence of the Bianchi identities, but we see here that it can be seen to arise from the fact that the Einstein tensor is locally conserved (and so the proposition may be extended in an obvious way).

The discussion here is still unnecessarily restrictive. Further examples arise from more general Riemannian functionals (e.g. Example 2.12), the use of generalised energy-momentum tensors and so forth. We conclude this section with some special cases.

4.1. **Local conformal invariants.** If $V(g)$ is a natural (scalar) conformal invariant of weight ℓ, meaning that $V(e^{2\omega}g) = e^{\ell\omega}V(g)$ then if $\ell \neq -n$ it is easily verified that $V(g)$ is naturally conformally variational, with (18) giving a functional. For example if W denotes the Weyl curvature then $|W|^2$ is a weight -4 conformal invariant, and so is conformally variational in dimensions greater than 4.

4.2. **Q-curvatures.** On Riemannian n-manifolds, there is an important class of natural scalar curvature quantities Q_m, parametrised by positive even integers m with $m \notin \{n, n+2, n+4, \ldots\}$, which are sometimes termed *subcritical Q-curvatures* [9]. In a conformal sense these generalise the scalar curvature: Q_2 is the scalar curvature ($n \geq 3$) and if $\widehat{g} = e^{2\omega}g$, $\omega \in C^\infty(M)$, then

$$Q_m^{\widehat{g}} = u^{\frac{n+m}{m-n}} \left(\delta S_m^g d + Q_m^g\right) u, \tag{25}$$

where, $u = e^{\frac{n-m}{2}\omega}$, $\delta = -\operatorname{div}$ is the formal adjoint of the exterior derivative d and S_m^g is an appropriate operator. The differential operator $P_m := \delta S_m^g d + Q_m^g$ is conformally invariant, and is $(\frac{2}{n-m}\times)$ the GJMS operator [27] with leading term the Laplacian power $\Delta^{m/2}$. Thus (25) is a higher order analogue of the Yamabe equation (which controls scalar curvature prescription). Considering now a curve $\widehat{g} = e^{2s\omega}g$, and differentiating at $s = 0$, we find

$$(Q_m)^\bullet = -mQ_m\omega + \frac{n-m}{2}\delta S_m^g d\omega.$$

It follows easily that Q_m is naturally conformally variational and arises from an action as given in Proposition 2.8 (with $\ell = -m$). Thus on closed manifolds the Q_m are constrained by (17) of Theorem 2.7.

The critical Q-curvature Q_n is a weight $-n$ Riemannian invariant on even n-manifolds, and is conformally variational [9, 13], although not known to be naturally so. Thus it satisfies the Kazdan-Warner type identity of Theorem 2.14 (and cf. [16] who first prove this and also the subcritical cases). In dimension 2 the critical Q curvature is the Gauss curvature and so is also covered by Theorem 2.11. It seems likely that the higher dimensional critical Q-curvatures could also be treated this way, but we shall not take that up here. An easy proof using conformal diffeomorphism invariance also follows from Theorem 7.1 of [12].

In summary: There are Q-curvatures Q_m for even integers $m \notin \{n+2, n+4, \ldots\}$ and the following holds.

Proposition 4.2. *For any conformal vector field X on a closed (M^n, g), $n \geq 2$, we have $\int_M \mathcal{L}_X Q_m \, dv_g = 0$.*

In dimensions $n \geq 2$, Q_2 is a non-zero multiple of the scalar curvature. Explicit formulae for the Q-curvatures Q_4, Q_6, and Q_8, as well as an algorithm for generating the higher Q_m, may be found in [21]. An alternative algorithm may be found in [26]. A recursive approach for the Q-curvature is developed in [32].

4.3. **Higher Einstein tensors.** Throughout the following we work on a manifold of dimension $n \geq 3$ and take $m \in 2\mathbb{Z}_{>0}$, with $m \notin \{n+2, n+4, \ldots\}$. With Q_m, as above and $\dim M = n \geq 3$, we define a class of natural tensors.

Definition 4.3. *Let $E^{(m)}$ be the symmetric natural 2-tensor defined by (7) (i.e. $E^{(m)} := B$) where*

$$\mathcal{S}(g) := (n-m)^{-1} \int_M Q_m^g dv_g, \quad \text{if} \quad m \neq n,$$

and $\mathcal{S}(g) := \int_M Q_m^g dv_g$, if $m = n$. Then we shall call $E^{(m)}$ a higher Einstein tensor.

The term "higher Einstein" is partly suggested by (25) and the following:

- For $m = 2$, and $n \geq 3$, $E^{(m)}$ is the usual Einstein tensor (up to a non-zero constant).
- Since each $E^{(m)}$ arises as a total metric variation we have
$$\nabla^a E_{ab}^{(m)} = 0,$$
as a special case of (9).
- Proposition 4.1 holds with $V = Q_m$ and $B^V = E^{(m)}$, with $m \neq n$.
- $Q_m = g^{ab} E_{ab}^{(m)}$, for $m \neq n$, and on Einstein manifolds Q_m is constant [19, 20].

Remark 4.4. *In the case of even manifolds M^n and $m = n$, $E^{(m)}$ is the Fefferman-Graham obstruction tensor of* [18], *see* [26]. *(In dimension $n = 4$ this is the well-known Bach tensor.) Thus in this case $E^{(m)}$ is trace-free and conformally invariant.*

The following is a special case of Theorem 3.1.

Proposition 4.5. *Let X be a conformal vector field on a compact manifold M with boundary ∂M. Then*

$$\int_N \mathcal{L}_X Q_m \, dv_g = -n \int_{\partial N} \overset{\circ}{E}{}_{ab}^{(m)} X^a \nu^b d\sigma_g, \tag{26}$$

for $m \neq n$, and where $\overset{\circ}{E}{}_{ab}^{(m)}$ is the trace-free part of $E_{ab}^{(m)}$.

Thus on even manifolds we may view the $E^{(m)}$ as "interpolating" between the usual Einstein tensor and the Fefferman-Graham obstruction tensor. The latter vanishes on Einstein manifolds [18, 22, 26]. These observations suggest an interesting problem:

Question: Do the $\overset{\circ}{E}{}_{ab}^{(m)}$ vanish on Einstein manifolds?

Since we posed this it has been observed by Graham that there a simple argument confirming that the answer is yes. So the higher Einstein tensors provide a strict weakening of the Einstein condition.

Theorem 4.6. *If* (M, g) *Einstein then* $\mathring{E}^{(m)}_{ab} = 0$ *for all* $m \in 2\mathbb{Z}_{>0}$*, with* $m \notin \{n+2, n+4, \ldots\}$.

Proof. [25] On any Riemannian manifold, the Q-curvatures may be given by formulae, the terms of which are simply complete metric contractions of covariant derivatives of the Ricci curvature, see Proposition 3.5 of [19], and the subsequent discussion there. On the other hand in (3.20) of the same source it is observed that a metric variation $h = dg^t/dt|_{t=0}$ induces a variation of the Ricci curvature which may be expressed purely in terms of covariant derivatives of h. Specifically:

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Ric}_{ij}(g^t) = \frac{1}{2}(\nabla^k\nabla_j h_{ik} + \nabla^k\nabla_i h_{jk} - \nabla^k\nabla_k h_{ij} - \nabla_i\nabla_j h^k{}_k).$$

The induced variation of the Levi-Civita takes a similar form

$$\frac{1}{2}g^{k\ell}(\nabla_j h_{i\ell} + \nabla_i h_{j\ell} - \nabla_\ell h_{ij}).$$

Putting these things together it follows easily that, on any Riemannian manifold, there is a formula for the $E^{(m)}_{ab}$ which is a linear combination of terms, each of which is a partial metric contraction of covariant derivatives of the Ricci curvature. Again no other curvature is involved in the formula. It follows easily that on an Einstein manifold $E^{(m)}_{ab}$ is simply a constant multiple of the metric. □

Remark 4.7. *Note that the constancy of the Q-curvatures on Einstein manifolds (mentioned earlier) is seen to be consistent with Theorem 4.6, by dint of Proposition 4.1, at least for* $m \neq n$*. (Of course to establish the result that the Q-curvatures are constant in this setting, including now the critical case, one would more easily use the first line of the proof of Theorem 4.6. The fact that one can argue this way was pointed out for the GJMS operators in the paragraph after Proposition 7.9 of* [19]*.)*

There is an analogue of Theorem 4.6, and its proof, for the gradient (as in (7)) of any natural scalar field arising as the restriction of a natural scalar on the Fefferman-Graham ambient manifold. In particular this applies to the $\mathring{B}^{(k)}_{ab}(g)$ *arising from the renormalised volume coefficients, as discussed in Proposition 4.9 below. These also vanish on Einstein manifolds.*

Remark 4.8. *In* [29] *Gursky makes several interesting remarks concerning the gradients* $E^{(m)}_{ab}$*. These are related to some of the ideas of Section 3.1. Surprisingly he is also able to define an analogue of* $\mathring{E}^{(4)}_{ab}$ *for conformally flat 4-manifolds, and (in this setting) this yields an identity of the form (26) for the critical Q-curvature. It would be interesting to investigate whether his tensor can be derived from a symmetry principle.*

4.4. **Renormalised volume coefficients.** Beginning with a manifold (M^n, g) $n \geq 3$, these natural scalar invariants v_k arise (see e.g. [23]) in the problem of [18] of finding a 1-parameter h_r of metrics, with $h_0 = g$ and so that

$$g_+ := \frac{dr^2 + h_r}{r^2}$$

is an asymptotic solution to $\mathrm{Ric}^{g_+} = -ng_+$ along $r = 0$ in $M_+ := M \times (0, \epsilon)$. The renormalised volume coefficients v_k are defined by a volume form expansion

$$\left(\frac{\det g_\rho}{\det g_0}\right) \sim 1 + \sum_{k=1}^{\infty} v_k \rho^k,$$

in the new variable $\rho = -\frac{1}{2}r^2$ with $g_\rho := h_r$. In odd dimension n this determines v_k for $k \in \mathbb{Z}_{\geq 1}$, but in even dimensions the mentioned formal problem is obstructed at finite order and so the v_k are in general defined for $k \in \{1, \cdots, \frac{n}{2}\}$ (but are defined for $k \in \mathbb{Z}_{\geq 1}$ in certain special cases, for example if g is Einstein or locally conformally flat).

Chang and Fang considered the $v_k(g)$ for the Yamabe type problem of conformally prescribing constant $v_k(g)$ [14]. They showed that for $n \neq 2k$ the equation $v_k(g)$ =constant is the Euler-Lagrange equation for the functional $\int_M v_k(g) dv_g$, under conformal variations satisfying the volume constraint $\int_M dv_g = 1$. This also follows from [24, Theorem 1.5] where Graham has shown that for $k \in \mathbb{Z}$, with $2k \leq n$ if n even, the infinitesimal conformal variation of the v_k takes the form

$$\frac{d}{dt} v_k(e^{2t\omega})|_{t=0} = -2k\omega v_k + \nabla_a(L^{ab}_{(k)} \nabla_b \omega), \tag{27}$$

with $L^{ab}_{(k)}$ a symmetric tensor (in fact more detail is given in [24]). It follows that, for $n \neq 2k$, the v_k are naturally conformally variational. Thus from Theorems 2.7, 3.1 and Corollary 3.3 we have immediately the following.

Proposition 4.9. *Let $k \in \mathbb{Z}$, with $2k < n$ if n even. The v_k satisfy Theorem 2.7. Moreover if we write $B^{(k)}_{ab}(g)$ for the gradient determined by (7) with $\mathcal{S}(g) := (n-2k)^{-1} \int_M v_k(g)\, dv_g$ then on any compact manifold N, of dimension $n \neq 2k$, with boundary ∂N, and with X a conformal vector field, we have*

$$\int_N \mathcal{L}_X v_k \, dv_g = -n \int_{\partial N} \mathring{B}^{(k)}_{ab} X^a \nu^b d\sigma_g,$$

where $\mathring{B}^{(k)}_{ab}$ is the trace-free part of $B^{(k)}_{ab}$.

For the v_k, with $2k < n$ if n even, this result specialises to Kazdan-Warner type identities via Corollary 3.2. Note that the differential operator on the right-hand-side of (27) is formally self-adjoint, so from [11, Lemma 2(ii)] (see Remark 2.15) this shows that the v_k are conformally variational, *including* $v_{n/2}$ for n even. Thus from Theorem 2.14, or equivalently [16, Theorem 2.1], we extend the above result as follows.

Proposition 4.10. *Let $n = 2k$, then for any conformal vector field X on a closed (M^n, g) we have*

$$\int_M \mathcal{L}_X v_k \, dv_g = 0.$$

Remark 4.11. *The Kazdan-Warner type identities for the v_k are first due to* [28]*. They use (27) and a specific calculation that follows the ideas of* [8]*. Our point is that, since (27) shows that the v_k are conformally variational, the results can also be deduced immediately from the general principles. Importantly using also the stronger fact that for $k \neq 2n$ the v_k are naturally conformally variational we also obtain the generalised Schoen-type identity of Proposition 4.9.*

For $k = 1, 2$, or when g is locally conformally flat, the v_k agree with the elementary symmetric functions $\sigma_k(g^{-1}P)$ of the Schouten tensor P, see [14, 24]*. So as noted in* [28] *the Kazdan-Warner type identities for v_k include also the similar results of Viaclovsky for the $\sigma_k(g^{-1}P)$,* [47].

4.5. **Gauss-Bonnet invariants and Einstein-Lovelock Tensors.** For $k \in \mathbb{Z}_{\geq 1}$ with $k \leq [n/2]$, the $2k$-Gauss-Bonnet curvature $S^{(2k)}$ is the complete contraction of the k^{th} tensor power of the Riemann curvature by the generalised Kronecker tensor, and has the property that in dimension $2k$ it is exactly the Pfaffian, i.e. the Chern-Gauss-Bonnet integrand (at least up to a nonzero constant). On (M, g) closed and Riemannian, with $\mathcal{S}^{(2k)}(g) := 2\int_M S^{(2k)}(g)\, dv_g$ the gradient $G^{(2k)}_{ab} = B_{ab}$ (in the sense of (7)) is called the Einstein-Lovelock tensor [37, 35] if $2k \neq n$. (If $2k = n$, then $\mathcal{S}^{(2k)}(g)$ is a multiple of the Euler characteristic.) Thus $G^{(2k)}_{ab}$ is locally conserved $\nabla^a G^{(2k)}_{ab} = 0$, for $2k \neq n$ $S^{(2k)}$ is naturally conformally variational, and as a special case of Theorem 3.1 we have the following.

Proposition 4.12. *Let X be a conformal vector field on a compact manifold M with boundary ∂M. Then for $2k \neq n$*

$$\int_N \mathcal{L}_X S^{(2k)} \, dv_g = -\frac{n}{2(n-2k)} \int_{\partial N} \mathring{G}^{(2k)}_{ab} X^a \nu^b \, d\sigma_g,$$

where $\mathring{G}^{(2k)}_{ab}$ is the trace-free part of $G^{(2k)}_{ab}$. In particular on closed manifolds $\int_N \mathcal{L}_X S^{(2k)} \, dv_g = 0$.

Remark 4.13. *The last conclusion giving a Kazdan-Warner type identity is also given by* [28] *using a direct calculation. Moreover they show that this also holds in the case $2k = n$. In fact it is easily verified that for k,such that the $S^{(2k)}$ are defined, the linearisation of the map $\omega \mapsto S^{(2k)}(e^{2\omega} g_0)$, $\omega \in C^\infty(M)$, is formally-self-adjoint. So that result may also be obtained from Theorem 2.14, or equivalently* [16, Theorem 2.1].

4.6. **Mean curvature of Euclidean hypersurfaces.** Let (M, g) be a codimension one submanifold of Euclidean space $\mathbb{E}^{n+1}$, with g_{ab} the pullback metric (i.e. the first fundamental form). We write ∇_a to denote the Levi-Civita connection of g. Let us write II_{ab} for the second fundamental form on M induced by the embedding. Then, since $\mathbb{E}^{n+1}$ is flat, II_{ab} satisfies the contracted Codazzi equation

(see e.g. [30])

$$\nabla^a II_{ab} - n\nabla_b H = 0, \tag{28}$$

where, as usual, g^{ab} is the inverse to g_{bc} and $H := \frac{1}{n} g^{cd} II_{cd}$ is the mean curvature of the embedding. Thus the symmetric 2-tensor

$$B_{ab} := II_{ab} - n g_{ab} H$$

is locally conserved everywhere on M: $\nabla^a B_{ab} = 0$. It follows immediately that Theorem 3.1 gives a Pohozaev-Schoen type identity on M (which we may take to have a boundary) with $V = n(1-n)H$. In particular as a special case of Corollary 3.2 we recover the following result.

Theorem 4.14. *Let $\iota : S^n \to \mathbb{E}^{n+1}$ be a conformal immersion with mean curvature H. Then for any conformal vector field X on S^n we have*

$$\int_{S^n} \mathcal{L}_X H dv_g = 0,$$

where we view H as a function on S^n, and dv_g is the pullback by ι of the first fundamental form measure.

This Theorem is first due to Ammann et al. [1]. There it is established using the fact that, on $\mathbb{E}^{n+1}$, the restriction of a parallel spinor to M satisfies a certain semilinear variant of the Dirac equation. They show that any spinor satisfying such an equation satisfies a Pohozaev-Schoen type identity. The argument above provides a direct alternative argument for the Kazdan-Warner type result in Theorem 4.14; in particular it avoids the use of spinor fields. The Pohozaev-Schoen identity of [1] is interesting and it would be interesting to investigate whether or not it is a special case of (21).

4.7. **The Pohozaev identity.** That the classical Pohozaev identity of [42],

$$\lambda n \int_M F(u) + \frac{2-n}{2}\lambda \int_M f(u)u = \frac{1}{2}\int_{\partial M} (x\cdot\nu)(\nabla_\nu u)^2,$$

follows from the identity of Schoen is stated in [45]. We have not been able to find the argument written anywhere so, for the convenience of the reader, and since it is an idea that generalises, we shall give here the derivation.

For any conformal vector field X on a Riemannian n-manifold (M, g) with smooth boundary ∂M, the following identity holds

$$\int_M \mathcal{L}_X \operatorname{Sc} \; dv_g = \frac{2n}{n-2}\int_{\partial M} \left(\operatorname{Ric} - \frac{1}{n}\operatorname{Sc}\cdot g\right)(X,\nu) d\sigma_g;$$

here ν is the outward normal, and Ric denotes the Ricci curvature.

We start by taking $M \subset \mathbb{R}^n$ with metric $g = u^{4/(n-2)} g_0$, g_0 the Euclidean metric (with $dvol^0_{n-1}$ on ∂M and $dvol^0_n$ on M resp.). Then with $p = \frac{n+2}{n-2}$ we have that

$$\operatorname{Sc} = -\frac{4(n-1)}{n-2} u^{-p} \Delta u$$

(Δ the Euclidean Laplacian) and we take the (Euler) conformal vector field $X = x_i \frac{\partial}{\partial x_i}$ (summation convention) to get

$$\mathcal{L}_X \,\mathrm{Sc} = -\frac{4(n-1)}{n-2}\left(x_i(-p)u^{-p-1}\frac{\partial u}{\partial x_i}\Delta u + u^{-p}x_i\frac{\partial}{\partial x_i}\Delta u\right)$$

and so with the relevant volumes, noting that $dvol_n = u^{p+1}dvol_n^0$ and $dvol_{n-1} = u^{2(n-1)/(n-2)}dvol_{n-1}^0$,

$$\mathcal{L}_X \,\mathrm{Sc}\, dvol_n = -\frac{4(n-1)}{n-2}\left(x_i(-p)\frac{\partial u}{\partial x_i}\Delta u + ux_i\frac{\partial}{\partial x_i}\Delta u\right) dvol_n^0$$

(from now on the volume forms will be understood as the Euclidean ones, and omitted). In a similar way we can find the boundary term, using $u^{p-1} = e^{2f}$ and

$$\mathrm{Ric} = (2-n)[\nabla df - df \otimes df] + [\Delta f - (n-2)|df|^2]g_0$$

which means that

$$\mathrm{Ric}_{ij} = (2-n)[\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}] + [\frac{\partial^2 f}{\partial x_k^2} - (n-2)\frac{\partial f}{\partial x_k}\frac{\partial f}{\partial x_k}]\delta_{ij}$$

and similarly in terms of u and its derivatives.

Now we first use the relation between the unit normals $\nu = u^{-2/(n-2)}\nu^0$, and then consider the identity in cases where u is very small along ∂M; finally taking the limiting case that $u = 0$ on ∂M. We find Schoen's identity then simplifies and determines the following relation:

$$-\frac{4(n-1)}{n-2}\int_M (-p)x_i\frac{\partial u}{\partial x_i}\Delta u + ux_i\frac{\partial}{\partial x_i}\Delta u =$$

$$\frac{2n}{n-2}\int_{\partial M}\Big((2-n)[-\frac{p-1}{2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - (\frac{p-1}{2})^2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}]+$$

$$[-\frac{p-1}{2}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_k} - (n-2)(\frac{p-1}{2})^2\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_k}]\delta_{ij}\Big)\nu_i^0 x_j$$

where as before $p - 1 = \frac{4}{n-2}$. Using the fact that ∇u is normal to the boundary we obtain

$$4(n-1)\frac{n+2}{n-2}\int_M x_i\frac{\partial u}{\partial x_i}\Delta u - 4(n-1)\int_M ux_i\frac{\partial}{\partial x_i}\Delta u$$

$$= 2n\frac{2n-2}{n-2}\int_{\partial M} u_\nu^2(\nu^0{\cdot}x).$$

Here we have used e.g.

$$\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\nu_i^0 x_j = u_\nu^2(\nu^0{\cdot}x).$$

Now the second integral over M may be integrated by parts (and no boundary term) to get the new integrand

$$-x_i\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} - u\Delta u$$

so we arrive at, after another integration by parts in the first term above, this time in the x_i variable,

$$4(n-1)\frac{n+2}{n-2}\int_M x_i\frac{\partial u}{\partial x_i}\Delta u - 4(n-1)\frac{n+2}{2}\int_M |\nabla u|^2$$
$$= 2\frac{n+2}{n-2}(n-1)\int_{\partial M} u_\nu^2(\nu^0\cdot x).$$

With (from the assumptions on u in the Pohozaev identity)

$$\int_M |\nabla u|^2 = \lambda\int_M uf(u)$$
$$\int_M x_i\frac{\partial u}{\partial x_i}\Delta u = n\lambda\int_M F(u)$$

we finally get

$$2n\lambda\int_M F(u) - (n-2)\lambda\int_M uf(u) = \int_{\partial M} u_\nu^2(\nu^0\cdot x)$$

which is the classical identity we wanted.

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